vecs.curl Curl, line integrals, and circulation

Line integrals

Consider a curve C in a Euclidean vector space $\mathbb{R}^3$. Let $\mathbf{r}(t) = [x(t), y(t), z(t)]$ be a parametric representation of C. Furthermore, let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector-valued function of $\mathbf{r}$ and let $\mathbf{r}'(t)$ be the tangent vector. The line integral is

line integral

$$\int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt \qquad (1)$$

which integrates $\mathbf{F}$ along the curve. For more on computing line integrals, see Schey[8] and Kreyszig.[9]

8. Schey, Div, Grad, Curl, and All that: An Informal Text on Vector Calculus, pp. 63-74.

9. Kreyszig, Advanced Engineering Mathematics, § 10.1, 10.2.

If $\mathbf{F}$ is a force being applied to an object moving along the curve C, the line integral is the work done by the force. More generally, the line integral integrates $\mathbf{F}$ along the tangent of C.

force
work

Circulation

Consider the line integral over a closed curve C, denoted by

$$\oint_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt. \qquad (2)$$

We call this quantity the circulation of $\mathbf{F}$ around C.

circulation

For certain vector-valued functions $\mathbf{F}$, the circulation is zero for every curve. In these cases (static electric fields, for instance), this is sometimes called the the law of circulation.

the law of circulation

Curl

Consider the division of the circulation around a curve in $\mathbb{R}^3$ by the surface area it encloses ΔS,

$$\frac{1}{\Delta S} \oint_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt. \qquad (3)$$

In a manner analogous to the operation that gaves us the divergence, let's consider shrinking this curve to a point and the surface area to zero,

$$\lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt. \tag{4}$$

We call this quantity the "scalar" curl of $\mathbf{F}$ at each point in $\mathbb{R}^3$ in the direction normal to ΔS as it shrinks to zero. Taking three (or n for $\mathbb{R}^n$) "scalar" curls in indepedent normal directions (enough to span the vector space), we obtain the curl proper, which is a vector-valued function $\operatorname{curl} : \mathbb{R}^3 \to \mathbb{R}^3$.

curl

curl

The curl is coordinate-independent. In cartesian coordinates, it can be shown to be equivalent to the following.

Equation 5 curl: differential form, cartesian coordinates

$$\operatorname{curl} \mathbf{F} = \begin{bmatrix} \partial_y F_z - \partial_z F_y & \partial_z F_x - \partial_x F_z & \partial_x F_y - \partial_y F_x \end{bmatrix}^\top$$

But what does the curl of $\mathbf{F}$ represent? It quantifies the local rotation of $\mathbf{F}$ about each point. If $\mathbf{F}$ represents a fluid's velocity, $\operatorname{curl} \mathbf{F}$ is the local rotation of the fluid about each point and it is called the vorticity.

vorticity

Zero curl, circulation, and path independence

Circulation

It can be shown that if the circulation of $\mathbf{F}$ on all curves is zero, then in each direction $\mathbf{n}$ and at every point $\operatorname{curl} \mathbf{F} = 0$ (i.e. $\mathbf{n} \cdot \operatorname{curl} \mathbf{F} = 0$). Conversely, for $\operatorname{curl} \mathbf{F} = 0$ in a simply connected region[10], $\mathbf{F}$ has zero circulation.

Succinctly, informally, and without the requisite

10. A region is simply connected if every curve in it can shrink to a point without leaving the region. An example of a region that is not simply connected is the surface of a toroid.

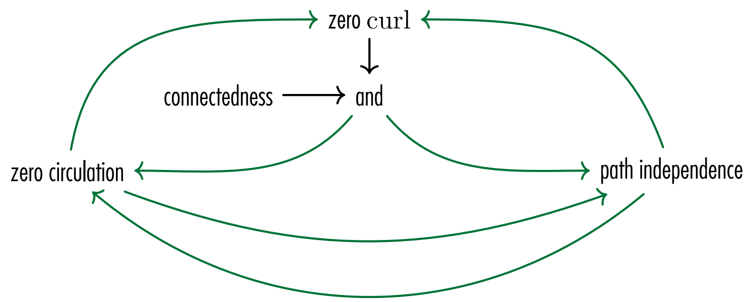

Figure curl.1: an implication graph relating zero curl, zero circulation, path independence, and connectedness. Green edges represent implication (a implies b) and black edges represent logical conjunctions.

qualifiers above,

$$\text{zero circulation} \Rightarrow \text{zero curl} \tag{6}$$

$$\text{zero curl} + \text{simply connected region} \Rightarrow \text{zero circulation.} \tag{7}$$

Path independence

It can be shown that if the path integral of $\mathbf{F}$ on all curves between any two points is path-independent, then in each direction n and at every point $\operatorname{curl} \mathbf{F} = 0$ (i.e. $n \cdot \operatorname{curl} \mathbf{F} = 0$). Conversely, for $\operatorname{curl} \mathbf{F} = 0$ in a simply connected region, all line integrals are independent of path. Succinctly, informally, and without the requisite qualifiers above,

path independence

$$\text{path independence} \Rightarrow \text{zero curl} \tag{8}$$

$$\text{zero curl} + \text{simply connected region} \Rightarrow \text{path independence.} \tag{9}$$

… and how they relate

It is also true that

$$\text{path independence} \Leftrightarrow \text{zero circulation.} \tag{10}$$

So, putting it all together, we get Fig. curl.1.

Exploring curl

Curl is perhaps best explored by considering it for a vector field in $\mathbb{R}^2$. Such a field in cartesian coordinates $\mathbf{F} = F_x \hat{\mathbf{i}} + F_y \hat{\mathbf{j}}$ has curl

$$\begin{aligned}\operatorname{curl}\mathbf{F} &= \begin{bmatrix}\partial_y 0 - \partial_z F_y & \partial_z F_x - \partial_x 0 & \partial_x F_y - \partial_y F_x\end{bmatrix}^\top \\ &= \begin{bmatrix}0 - 0 & 0 - 0 & \partial_x F_y - \partial_y F_x\end{bmatrix}^\top \\ &= \begin{bmatrix}0 & 0 & \partial_x F_y - \partial_y F_x\end{bmatrix}^\top . \qquad (11)\end{aligned}$$

That is, $\operatorname{curl}\mathbf{F} = (\partial_x F_y - \partial_y F_x)\hat{\mathbf{k}}$ and the only nonzero component is normal to the xy-plane. If we overlay a quiver plot of $\mathbf{F}$ over a "color density" plot representing the $\hat{\mathbf{k}}$-component of $\operatorname{curl}\mathbf{F}$, we can increase our intuition about the curl.

The following was generated from a Jupyter notebook with the following filename and kernel.

```
notebook filename: curl-and-line-integrals.ipynb
notebook kernel: python3
```

First, load some Python packages.

```
from sympy import *
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.ticker import LogLocator
from matplotlib.colors import *
from sympy.utilities.lambdify import lambdify
from IPython.display import display, Markdown, Latex
```

Now we define some symbolic variables and functions.

```
var('x,y')
F_x = Function('F_x')(x,y)
F_y = Function('F_y')(x,y)
```

We use the same function defined in Lec. vecs.div, `quiver_plotter_2D`, to make several of these plots.

Let's inspect several cases.

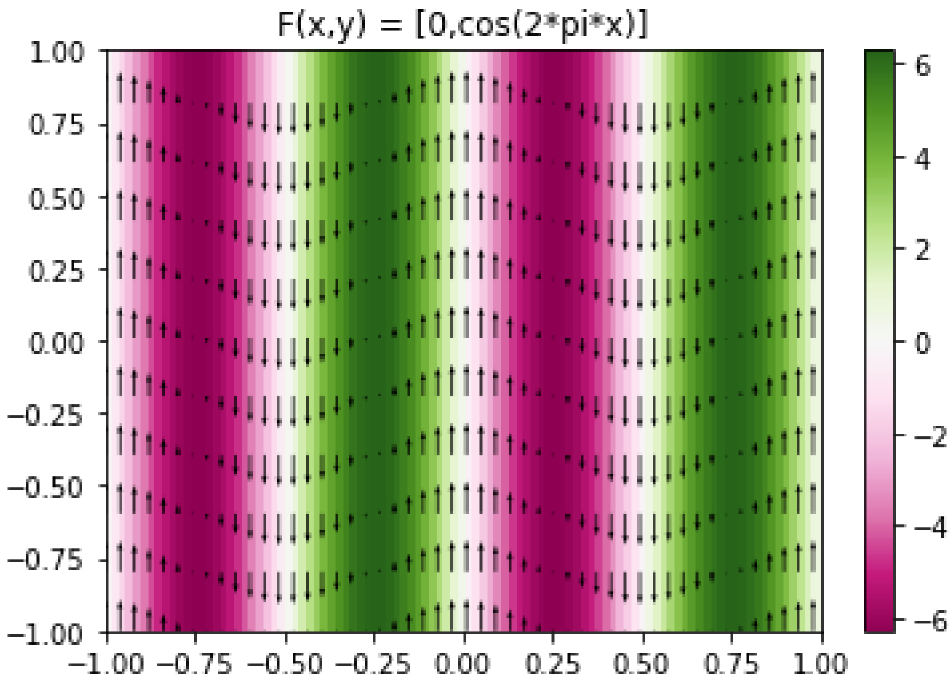

Figure curl.2: png

```
p = quiver_plotter_2D(
  field={F_x:0,F_y:cos(2*pi*x)},
  density_operation='curl',
  grid_decimate_x=2,
  grid_decimate_y=10,
  grid_width=1
)
```

```
The curl is:
```

$-2\pi\sin(2\pi x)$

```
p = quiver_plotter_2D(
  field={F_x:0,F_y:x**2},
  density_operation='curl',
  grid_decimate_x=2,
  grid_decimate_y=20,
)
```

```
The curl is:
```

$2x$

```
p = quiver_plotter_2D(
  field={F_x:y**2,F_y:x**2},
```

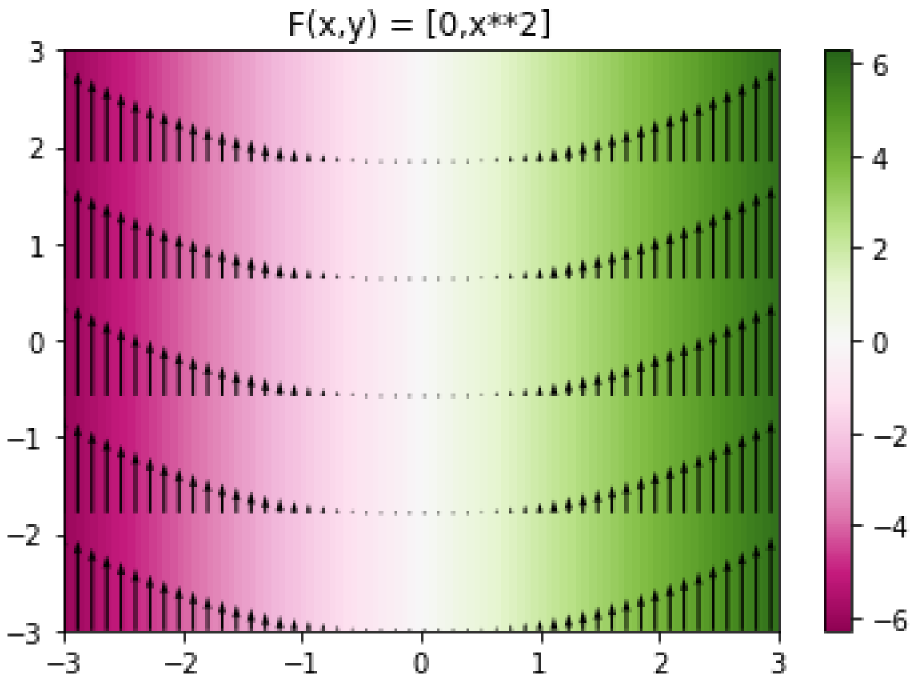

Figure curl.3: png

```
  density_operation='curl',
  grid_decimate_x=2,
  grid_decimate_y=20,
)
```

```
The curl is:
```

$2x - 2y$

```
p = quiver_plotter_2D(
  field={F_x:-y,F_y:x},
  density_operation='curl',
  grid_decimate_x=6,
  grid_decimate_y=6,
)
```

```
Warning: density operator is constant (no density
↪  plot)
The curl is:
```

2

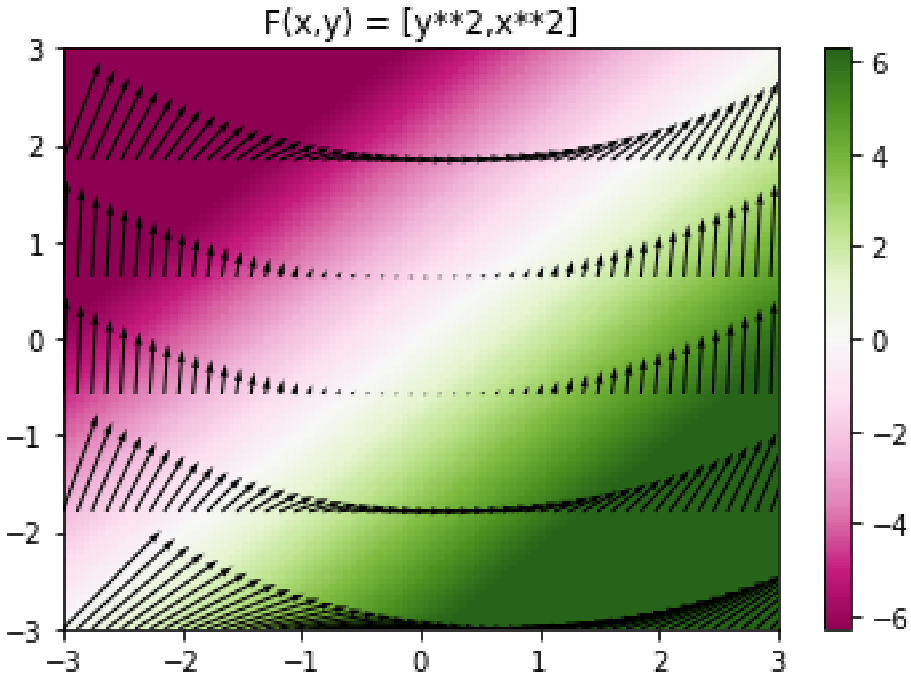

Figure curl.4: png

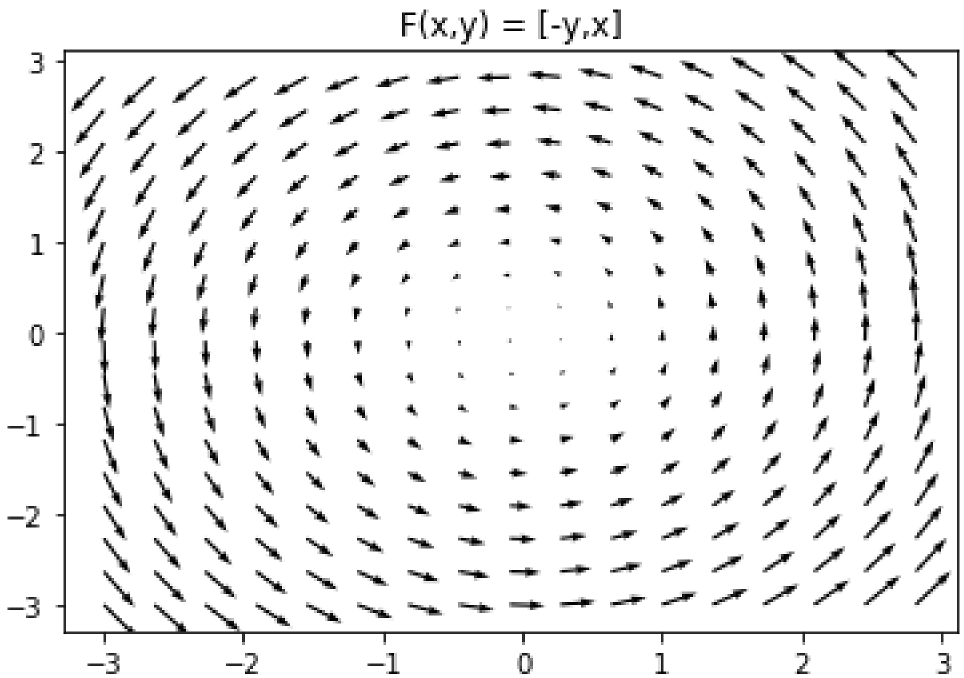

Figure curl.5: png