# vecs.stoked Stokes and divergence theorems

Two theorems allow us to exchange certain integrals in $\mathbb{R}^3$ for others that are easier to evaluate.

## The divergence theorem

The divergence theorem asserts the equality of the surface integral of a vector field $\mathbf{F}$ and the triple integral of $\operatorname{div}\mathbf{F}$ over the volume $V$ enclosed by the surface $S$ in $\mathbb{R}^3$. That is,

**divergence theorem**

**triple integral**

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_V \operatorname{div}\mathbf{F}\,dV. \qquad (1)$$

Caveats are that $V$ is a closed region bounded by the orientable[12] surface $S$ and that $\mathbf{F}$ is continuous and continuously differentiable over a region containing $V$. This theorem makes some intuitive sense: we can think of the divergence inside the volume "accumulating" via the triple integration and equaling the corresponding surface integral. For more on the divergence theorem, see Kreyszig[13] and Schey.[14] A lovely application of the divergence theorem is that, for any quantity of conserved stuff (mass, charge, spin, etc.) distributed in a spatial $\mathbb{R}^3$ with time-dependent density $\rho : \mathbb{R}^4 \to \mathbb{R}$ and velocity field $\boldsymbol{v} : \mathbb{R}^4 \to \mathbb{R}^3$, the divergence theorem can be applied to find that

**orientable**

12. A surface is orientable if a consistent normal direction can be defined. Most surfaces are orientable, but some, notably the Möbius strip, cannot be. See Kreyszig (Kreyszig, Advanced Engineering Mathematics, § 10.6) for more.

13. ibidem, § 10.7.

14. Schey, Div, Grad, Curl, and All that: An Informal Text on Vector Calculus, pp. 45-52.

$$\partial_t\rho = -\operatorname{div}(\rho\boldsymbol{v}), \qquad (2)$$

which is a more general form of a continuity equation, one of the governing equations of many physical phenomena. For a derivation of this equation, see Schey.[15]

**continuity equation**

15. ibidem, pp. 49-52.

## The Kelvin-Stokes' theorem

The Kelvin-Stokes' theorem asserts the equality of the circulation of a vector field $\mathbf{F}$ over a closed curve $C$ and the surface integral of $\operatorname{curl}\mathbf{F}$ over a

**Kelvin–Stokes' theorem**

surface $S$ that has boundary $C$. That is, for $\mathbf{r}(t)$ a parameterization of $C$ and surface normal $\mathbf{n}$,

$$\oint_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \iint_S \mathbf{n} \cdot \operatorname{curl} \mathbf{F}\,dS. \tag{3}$$

Caveats are that $S$ is piecewise smooth,[16] its boundary $C$ is a piecewise smooth simple closed curve, and $\mathbf{F}$ is continuous and continuously differentiable over a region containing $S$. This theorem is also somewhat intuitive: we can think of the divergence over the surface “accumulating” via the surface integration and equaling the corresponding circulation. For more on the Kelvin-Stokes’ theorem, see Kreyszig[17] and Schey.[18]

**piecewise smooth**

16. A surface is smooth if its normal is continuous everywhere. It is piecewise smooth if it is composed of a finite number of smooth surfaces.

17. Kreyszig, Advanced Engineering Mathematics, § 10.9.

18. Schey, Div, Grad, Curl, and All that: An Informal Text on Vector Calculus, pp. 93-102.

## Related theorems

Greene’s theorem is a two-dimensional special case of the Kelvin-Stokes’ theorem. It is described by Kreyszig.[19]

**Greene’s theorem**

19. Kreyszig, Advanced Engineering Mathematics, § 10.9.

It turns out that all of the above theorems (and the fundamental theorem of calculus, which relates the derivative and integral) are special cases of the generalized Stokes’ theorem defined by differential geometry. We would need a deeper understanding of differential geometry to understand this theorem. For more, see Lee.[20]

**generalized Stokes’ theorem**

20. Lee, Introduction to Smooth Manifolds, Ch. 16.