four.general Generalized fourier series and orthogonality

Let $f : \mathbb{R} \to \mathbb{C}$, $g : \mathbb{R} \to \mathbb{C}$, and $w : \mathbb{R} \to \mathbb{C}$ be complex functions. For square-integrable[2] f, g, and w, the inner product of f and g with weight function w over the interval $[a, b] \subseteq \mathbb{R}$ is[3]

inner product

weight function

$$\langle f, g\rangle_w = \int_a^b f(x)\overline{g}(x)w(x)\,dx \tag{1}$$

where $\overline{g}$ denotes the complex conjugate of g. The inner product of functions can be considered analogous to the inner (or dot) product of vectors.

The fourier series components can be found by a special property of the $\sin$ and $\cos$ functions called orthogonality. In general, functions f and g from above are orthogonal over the interval $[a, b]$ iff

orthogonality

$$\langle f, g\rangle_w = 0 \tag{2}$$

for weight function w. Similar to how a set of orthogonal vectors can be a basis for a vector space, a set of orthogonal functions can be a basis for a function space: a vector space of functions from one set to another (with certain caveats).

basis

function space

In addition to some sets of sinusoids, there are several other important sets of functions that are orthogonal. For instance, sets of legendre polynomials (Erwin Kreyszig. Advanced Engineering Mathematics. 10th. John Wiley & Sons, Limited, 2011. ISBN: 9781119571094. The authoritative resource for engineering mathematics. It includes detailed accounts of probability, statistics, vector calculus, linear algebra, fourier analysis, ordinary and partial differential equations, and complex analysis. It also includes several other topics with varying degrees of depth. Overall, it is the best place to start when seeking mathematical guidance.

legendre polynomials

2. A function f is square-integrable if $\int_{-\infty}^{\infty} |f(x)|^2\,dx < \infty$.

3. This definition of the inner product can be extended to functions on $\mathbb{R}^2$ and $\mathbb{R}^3$ domains using double- and triple-integration. See (Schey, Div, Grad, Curl, and All that: An Informal Text on Vector Calculus, p. 261).

§ 5.2) and bessel functions (Kreyszig, Advanced Engineering Mathematics, § 5.4) are orthogonal. As with sinusoids, the orthogonality of some sets of functions allows us to compute their series components. Let functions $f_0, f_1, \cdots$ be orthogonal with respect to weight function w on interval $[a, b]$ and let $\alpha_0, \alpha_1, \cdots$ be real constants. A generalized fourier series is (ibidem, § 11.6)

bessel functions

generalized fourier series

$$f(x) = \sum_{m=0}^{\infty} \alpha_m f_m(x) \tag{3}$$

and represents a function f as a convergent series. It can be shown that the Fourier components α_m can be computed from

Fourier components

$$\alpha_m = \frac{\langle f, f_m \rangle_w}{\langle f_m, f_m \rangle_w}. \tag{4}$$

In keeping with our previous terminology for fourier series, Eq. 3 and Eq. 4 are called general fourier synthesis and analysis, respectively. For the aforementioned legendre and bessel functions, the generalized fourier series are called fourier-legendre and fourier-bessel series (ibidem, § 11.6). These and the standard fourier series (Lec. four.series) are of particular interest for the solution of partial differential equations (Chapter pde).

synthesis

analysis

fourier−legendre series

fourier−bessel series