pde.class Classifying PDEs

PDEs often have an infinite number of solutions; however, when applying them to physical systems, we usually assume there is a deterministic or at least a probabilistic sequence of events will occur. Therefore, we impose additonal constraints on a PDE usually in the form of

initial conditions

1. initial conditions, values of independent variables over all space at an initial time and

boundary conditions

2. boundary conditions, values of independent variables (or their derivatives) over all time.

Ideally, imposing such conditions leaves us with a well-posed problem, which has three aspects. (Antonio Bove, F. (Ferruccio) Colombini and Daniele Del Santo. Phase space analysis of partial differential equations. eng. Progress in nonlinear differential equations and their applications ; v. 69. Boston ; Berlin: Birkhäuser, 2006. ISBN: 9780817645212, § 1.5)

well–posed problem

existence There exists at least one solution.

uniqueness There exists at most one solution.

stability If the PDE, boundary conditons, or initial conditions are changed slightly, the solution changes only slightly.

linear
nonlinear

As with ODEs, PDEs can be linear or nonlinear; that is, the independent variables and their derivatives can appear in only linear combinations (linear PDE) or in one or more nonlinear combination (nonlinear PDE). As with ODEs, there are more known analytic solutions to linear PDEs than nonlinear PDEs.

order

The order of a PDE is the order of its highest partial derivative. A great many physical models can be described by second-order PDEs or systems thereof. Let u be an independent

second–order PDEs

scalar variable, a function of m temporal and spatial variables $x_i \in \mathbb{R}^n$. A second-order linear PDE has the form, for coefficients $\alpha, \beta, \gamma, \delta$, real functions of x_i, (W.A. Strauss. Partial Differential Equations: An Introduction. Wiley, 2007. ISBN: 9780470054567. A thorough and yet relatively compact introduction. § 1.6)

$$\underbrace{\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}\partial^2_{x_i x_j}u +}_{\text{second-order terms}} \underbrace{\sum_{k=1}^{m}\left(\gamma_k \partial_{x_k} u + \delta_k u\right)}_{\text{first- and zeroth-order terms}} = \underbrace{f(x_1, \cdots, x_n)}_{\text{forcing}} \tag{1}$$

where f is called a forcing function. When f is zero, Eq. 1 is called homogeneous. We can consider the coefficients α_{ij} to be components of a matrix A with rows indexed by i and columns indexed by j. There are four prominent classes defined by the eigenvalues of A:

forcing function

homogeneous

elliptic the eigenvalues all have the same sign,

parabolic the eigenvalues have the same sign except one that is zero,

hyperbolic exactly one eigenvalue has the opposite sign of the others, and

ultrahyperbolic at least two eigenvalues of each signs.

The first three of these have received extensive treatment. They are named after conic sections due to the similarity the equations have with polynomials when derivatives are considered analogous to powers of polynomial variables. For instance, here is a case of each of the first three classes,

$$\partial^2_{xx}u + \partial^2_{yy}u = 0 \qquad \text{(elliptic)}$$
$$\partial^2_{xx}u - \partial^2_{yy}u = 0 \qquad \text{(hyperbolic)}$$
$$\partial^2_{xx}u - \partial_t u = 0. \qquad \text{(parabolic)}$$

When A depends on x_i, it may have multiple classes across its domain. In general, this equation and its associated initial and boundary conditions do not comprise a well-posed problem; however several special cases have been shown to be well-posed. Thus far, the most general statement of existence and uniqueness is the cauchy-kowalevski theorem for cauchy problems.

cauchy–kowalevski theorem
cauchy problems